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Rational Approximations from Power Series of Vector-Valued Meromorphic Functions*

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Let $F(z)$ be a vector-valued function, $F: C \to C^N$, which is analytic at $z=0$ and meromorphic in a neighbourhood of $z=0$, and let its Maclaurin series be given. In this work we develop vector-valued rational approximation procedures for $F(z)$ by applying vector extrapolation methods to the sequence of partial sums of its Maclaurin series. We analyze some of the algebraic and analytic properties of the rational approximations thus obtained and show that they are akin to Padé approximants. In particular, we prove a Koenig-type theorem concerning their poles and a de Montessus-type theorem concerning their uniform convergence. We show how "optimal" approximations to multiple poles and to Laurent expansions about these poles can be constructed. Extensions of the procedures above and the accompanying theoretical results to functions defined in arbitrary linear spaces is also considered. One of the most interesting and immediate applications of the results of this work is to the matrix eigenvalue problem. In a new work we exploit the developments of the present work to devise bona fide generalizations of the classical power method that are especially suitable for very large and sparse matrices. These generalizations can be used to approximate simultaneously several of the largest distinct eigenvalues and corresponding eigenvectors and invariant subspaces of arbitrary matrices, which may or may not be diagonalizable, and are very closely related with known Krylov subspace methods.

1. INTRODUCTION

Let $F(z)$ be a vector-valued function, $F: C \to C^N$, which is analytic at $z=0$ and meromorphic in a neighbourhood of $z=0$, and let its Maclaurin series be given as

$$F(z) = \sum_{m=0}^{\infty} u_m z^m, \tag{1.1}$$

where u_m are fixed vectors in C^N.

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In this work we propose three types of vector-valued rational approximation procedures that are based entirely on the expansion in (1.1). For each of these procedures the rational approximations have two indices, n and k, attached to them, and thus form a two-dimensional table akin to the Padé table or the Walsh array. Let us denote the (n, k) entry of this table by $F_{n,k}(z)$. Then $F_{n,k}(z)$, if it exists, is defined to be of the form

$$F_{n,k}(z) = \frac{\sum_{j=0}^{k} c_j^{(n,k)} z^{k-j} F_{n+\nu+j}(z)}{\sum_{j=0}^{k} c_j^{(n,k)} z^{k-j}} \equiv \frac{P_{n,k}(z)}{Q_{n,k}(z)} \quad \text{with} \quad c_k^{(n,k)} = Q_{n,k}(0) = 1, \tag{1.2}$$

where ν is an arbitrary but otherwise fixed integer $\geqslant -1$,

$$F_m(z) = \sum_{i=0}^{m} u_i z^i, \quad m = 0, 1, 2, ...; \qquad F_m(z) \equiv 0 \quad \text{for} \quad m < 0, \tag{1.3}$$

and the $c_j^{(n,k)}$ are scalars that depend on the approximation procedure being used.

If we denote the three approximation procedures by SMPE, SMMPE, and STEA (the source of these abbreviations is discused later in this section), then the $c_j^{(n,k)} \equiv c_j$, for each of the three procedures, satisfy a linear system of equations of the form

$$\sum_{j=0}^{k-1} u_{ij} c_j = -u_{ik}, \qquad 0 \leqslant i \leqslant k-1; \quad c_k = 1, \tag{1.4}$$

where u_{ij} are scalars defined as

$$u_{ij} = \begin{cases} (u_{n+i}, u_{n+j}) & \text{for SMPE,} \\ (q_{i+1}, u_{n+j}) & \text{for SMMPE,} \\ (q, u_{n+i+j}) & \text{for STEA.} \end{cases} \tag{1.5}$$

Here $(\cdot, \cdot)$ is an inner product—not necessarily the standard Euclidean inner product—whose homogeneity property is such that $(\alpha x, \beta y) = \bar{\alpha}\beta(x, y)$ for $x, y \in C^N$ and $\alpha, \beta \in C$. The vectors $q_1, q_2, ...$, form a linearly independent set, and the vector q is nonzero. Obviously, $F_{n,k}(z)$ exists if the linear system in (1.4) has a solution for $c_0, c_1, ..., c_{k-1}$.

Note that the rational approximations above can be defined for any vector-valued function that is analytic at $z = 0$ regardless of whether it is meromorphic or not. All that is needed to construct these approximations is the Maclaurin series of the function being considered.

It is easy to verify that for SMPE the equations in (1.4) involving $c_0, c_1, ..., c_{k-1}$ are the normal equations for the least-squares problem

$$\min_{c_0, c_1, ..., c_{k-1}} \left\| \sum_{j=0}^{k-1} c_j u_{n+j} + u_{n+k} \right\|, \tag{1.6}$$

where the norm $\|\cdot\|$ is that induced by the inner product $(\cdot, \cdot)$, namely, $\|x\| = \sqrt{(x, x)}$.

As can be seen from (1.4) and (1.5), the denominator polynomial $Q_{n,k}(z)$ is constructed from $u_n, u_{n+1}, ..., u_{n+k}$ for SMPE and SMMPE and from $u_n, u_{n+1}, ..., u_{n+2k-1}$ for STEA. Once the denominators have been determined, the numerators involve $u_0, u_1, ..., u_{n+\nu+k}$ for all three approximation procedures. That is to say, the approximations $F_{n,k}(z)$ are constructed from a finite number of terms of the Maclaurin series of $F(z)$.

The approximation procedures above are very closely related to some vector extrapolation methods. In fact, as is stated in Theorem 2.3 in Section 2 of the present work, $F_{n,k}(z)$ for SMPE, SMMPE, and STEA are obtained by applying some generalized versions of the minimal polynomial extrapolation (MPE), the modified minimal polynomial extrapolation (MMPE), and the topological epsilon algorithm (TEA), respectively, to the vector sequence $F_m(z)$, $m = 0, 1, 2, ...$ These are methods for accelerating the convergence of vector sequences such as those that arise from fixed-point iterative techniques for linear or nonlinear systems of equations. For early references pertaining to these methods and their description see the survey paper of Smith *et al.* [SmFSi]. For their convergence properties and other recent developments see the papers by Sidi [Si1, Si2, Si5], Sidi and Bridger [SiB], Sidi *et al.* [SiFSm], and Ford and Sidi [FSi]. The above-mentioned generalizations of vector extrapolation methods are given in [SiB, Eqs. (1.16) and (1.17)].

In Theorems 2.1 and 2.2 in Section 2 we show that the approximations $F_{n,k}(z)$ enjoy some Padé-like properties.

In Section 3 we give some simple technical results concerning the structure of the u_m and $F_m(z)$, $m = 0, 1, 2, ...,$ when the function $F(z)$ is meromorphic in a disk, and we also introduce some conditions on $F(z)$ and on the procedures SMPE, SMMPE, and STEA, which seem to be necessary to obtain the main results of Section 4.

One of the main aims of this work is to present a detailed analysis of the approximations $F_{n,k}(z)$ that have been defined above, for $n \to \infty$. In Section 4 we start by proving a Koenig-type theorem for the denominator polynomials $Q_{n,k}(z)$. In particular, we analyze the convergence behavior of these polynomials and prove that their zeros tend to the k smallest poles of $F(z)$, counted according to their multiplicities, under certain conditions, providing at the same time precise rates of convergence for them. All this is done in Theorems 4.1 and 4.5. We next analyze the convergence of the $F_{n,k}(z)$ in the complex z-plane and prove a de Montessus-type theorem on their uniform convergence. This is done in Theorem 4.2. Other useful results pertaining to the $F_{n,k}(z)$ and their poles and corresponding residues are given in Theorems 4.3 and 4.4 and in Section 5.

In Section 6 we extend all the developments of the previous sections to general linear spaces under appropriate assumptions.

It turns out that the denominator polynomials $Q_{n,k}(z)$ are very closely related to some recent extensions of the power method for the matrix eigenvalue problem, see [SiB, Section 6; Si3]. Specifically, if the vectors u_m of (1.1) are obtained from $u_m = Au_{m-1}$, $m = 1, 2, \ldots$, with u_0 abitrary and A being a complex $N \times N$ and, in general, nondiagonalizable matrix, then the reciprocals of the zeros of the polynomial $Q_{n,k}(z)$ are approximations to the largest distinct and, in general, defective eigenvalues of A under certain conditions. In a recent paper [Si6], we provide precise error bounds for these approximations based on the results of Theorems 4.1 and 4.5, where we also extend the treatment of [SiB, Si3] to cover eigenvectors and invariant subspaces. Again, in the same paper, we explore the connection of this new approach with Krylov subspace methods.

The techniques that we use in this work are those that were developed in [Si1, Si3, SiB, SiFSm], and the recent work of Sidi [Si4] on classical Padé approximants. In particular, the treatment of the matrix eigenvalue problem was motivated by the developments of [Si4].

The subject of rational approximations to vector-valued functions has received considerable attention lately. We mention some of the recent literature dealing with functions $F(z)$ that are defined by their Maclaurin series (1.1). In [G] Graves-Morris developed the generalized inverse vector-valued Padé approximants and showed that they are also obtained by applying the vector epsilon algorithm to the vector sequence $F_m(z)$, $m = 0, 1, 2, \ldots$. This work was later extended by Graves-Morris and Jenkins in [GJ1, GJ2]. Determinantal representations for these rational approximations were provided in [GJ2]. In [GSa] Graves-Morris and Saff analyzed in great detail the convergence behavior of these approximations for functions $F(z)$ that are meromorphic in a neighbourhood of the origin and gave some uniform convergence theorems of de Montessus type. For details and additional related references we refer the reader to [GSa].

2. Padé-Like Properties

From (1.2) and (1.3) it is obvious that the numerator $P_{n,k}(z)$ of $F_{n,k}(z)$ is a vector-valued polynomial of degree $\leqslant n + \nu + k$, while its denominator $Q_{n,k}(z)$ is a scalar-valued polynomial of degree at most k.

The special structure of $F_{n,k}(z)$ immediately suggests the following Padé-like property:

THEOREM 2.1. *If it exists, $F_{n,k}(z)$ satisfies*

$$F(z) - F_{n,k}(z) = O(z^{n+\nu+k+1}) \quad \text{as} \quad z \to 0. \tag{2.1}$$

Proof. From (1.2) we have

$$Q_{n,k}(z)\,F(z) - P_{n,k}(z) = \sum_{j=0}^{k} c_j z^{k-j}[F(z) - F_{n+\nu+j}(z)]. \tag{2.2}$$

The result now follows by realizing that $F(z) - F_m(z) = O(z^{m+1})$ as $z \to 0$, $m = 0, 1, \ldots$. ∎

The property contained in (2.1) is Padé-like in the sense that, for SMPE and SMMPE with $\nu = 0$, $F_{n,k}(z)$ is constructed using $u_0, u_1, \ldots, u_{n+k}$, and $F(z) - F_{n,k}(z) = O(z^{n+k+1})$ as $z \to 0$, while for STEA with $\nu = k-1$, $F_{n,k}(z)$ is constructed using $u_0, u_1, \ldots, u_{n+2k-1}$, and $F(z) - F_{n,k}(z) = O(z^{n+2k})$ as $z \to 0$.

Note that the Padé-like property in (2.1) is a result of (1.2) only, and it does not depend on how the c_j are determined. As such, it cannot be a factor in determining the true behavior of $F_{n,k}(z)$ as an approximation to $F(z)$. It is the linear system in (1.4) that determines the behavior of $F_{n,k}(z)$, as we see in the next sections.

Using the linear system in (1.4), we can derive a determinant representation for $F_{n,k}(z)$ that resembles the known representation for Padé approximants.

THEOREM 2.2. *$F_{n,k}(z)$ has the determinant representation*

$$F_{n,k}(z) = \frac{D(z^k F_{n+\nu}(z), z^{k-1} F_{n+\nu+1}(z), \ldots, z^0 F_{n+\nu+k}(z))}{D(z^k, z^{k-1}, \ldots, z^0)}, \tag{2.3}$$

where $D(\sigma_0, \sigma_1, \ldots, \sigma_k)$ is the determinant

$$D(\sigma_0, \sigma_1, \ldots, \sigma_k) = \begin{vmatrix} \sigma_0 & \sigma_1 & \cdots & \sigma_k \\ u_{00} & u_{01} & \cdots & u_{0k} \\ u_{10} & u_{11} & \cdots & u_{1k} \\ \vdots & \vdots & & \vdots \\ u_{k-1,0} & u_{k-1,1} & \cdots & u_{k-1,k} \end{vmatrix}. \tag{2.4}$$

In case σ_i are vectors, we interpret (2.4) *as*

$$D(\sigma_0, \sigma_1, \ldots, \sigma_k) = \sum_{i=0}^{k} \sigma_i N_i, \tag{2.5}$$

N_i being the cofactor of σ_i in (2.4).

Proof. Left to the reader. ∎

We note that the determinant representation above serves as a very useful tool in the analysis of $F_{n,k}(z)$.

Finally, the next result sums up the connection between $F_{n,k}(z)$ and the various vector extrapolation methods.

THEOREM 2.3. *The approximations $F_{n,k}(z)$ for the three procedures SMPE, SMMPE, and STEA are obtained by applying the generalized versions of MPE, MMPE, and TEA, respectively, as they are given in* [SiB], *to the vector sequence $F_m(z)$, $m = 0, 1, 2, \ldots$.*

Proof. By performing elementary row and column transformations on the determinant representations given in [SiB, Eq. (1.17)], we obtain (2.3). We leave the details to the reader. ∎

3. TECHNICAL PRELIMINARIES AND ASSUMPTIONS

Assume now that the vector-valued function $F(z)$ is analytic at $z = 0$ and meromorphic in the open disk $K = \{z : |z| < R\}$. Let $z_j \equiv \lambda_j^{-1}$, $j = 1, 2, \ldots, t$, be the distinct poles of $F(z)$ in K, whose respective multiplicities are $p_j + 1 \equiv \omega_j$, $j = 1, 2, \ldots, t$. Let the z_j be ordered such that

$$0 < |z_1| \leqslant |z_2| \leqslant \cdots \leqslant |z_t| < R, \tag{3.1}$$

which implies the ordering

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_t| > R^{-1}. \tag{3.2}$$

Consequently, $F(z)$ has the representation

$$F(z) = \sum_{j=1}^{t} \sum_{i=0}^{p_j} \frac{a_{ji}}{(1 - \lambda_j z)^{i+1}} + G(z), \tag{3.3}$$

where a_{ji} are constant vectors in C^N, $a_{jp_j} \neq 0$, $1 \leqslant j \leqslant t$, and $G(z)$ is analytic in K and thus has the convergent expansion

$$G(z) = \sum_{j=0}^{\infty} g_j z^j, \qquad z \in K. \tag{3.4}$$

LEMMA 3.1. *The coefficients u_m of the power series of $F(z)$ in (1.1) are given by*

$$u_m = \sum_{j=1}^{t} \left[\sum_{l=0}^{p_j} \tilde{a}_{jl} \binom{m}{l} \right] \lambda_j^m + \tilde{a}(m, \xi)\, \xi^m, \tag{3.5}$$

where $\tilde{a}_{jl}$ are constant vectors in C^N defined as

$$\tilde{a}_{jl} = \sum_{i=l}^{p_j} a_{ji} \binom{i}{i-l}, \tag{3.6}$$

and the vectors $\tilde{a}(m, \xi) = (a_1(m, \xi), ..., a_N(m, \xi))^T$ are such that $\tilde{a}(m, \xi)\xi^m = g_m$; thus, for each i,

$$|a_i(m, \xi)| \leqslant M_i(\xi) \equiv \max_{|z| = \xi^{-1}} |G_i(z)|, \quad \xi^{-1} \in (|z_t|, R), \qquad \textit{but arbitrary.} \tag{3.7}$$

Here $G_i(z)$ stands for the ith component of $G(z)$.

Proof. The proof can be achieved by applying Lemma 4.1 of [Si4] to each component of u_m. ▮

LEMMA 3.2. *The partial sums $F_m(z)$ in (1.3) satisfy*

$$F(z) - F_m(z) = \sum_{j=1}^{t} \left[\sum_{l=0}^{p_j} b_{jl}(z) \binom{m}{l} \right] (\lambda_j z)^m + b(m, \xi, z)(\xi z)^m, \tag{3.8}$$

where $b_{jl}(z)$ are vector-valued rational functions of z defined as

$$b_{jl}(z) = \sum_{i=l}^{p_j} a_{ji} \sum_{h=l}^{i} \binom{i+1}{h-l} \left(\frac{\lambda_j z}{1 - \lambda_j z} \right)^{i-h+1}, \tag{3.9}$$

and the vectors $b(m, \xi, z) = (b_1(m, \xi, z), ..., b_N(m, \xi, z))^T$ are such that, for each i,

$$|b_i(m, \xi, z)| \leqslant M_i(\xi) \frac{|\xi z|}{1 - |\xi z|}, \qquad |z| < \xi^{-1}, \tag{3.10}$$

$M_i(\xi)$ and ξ being as in (3.7).

Proof. The proof can be achieved by applying Lemma 5.1 of [Si4] to each component of $F_m(z)$. ▮

In the sequel we asume that for the approximation procedures SMPE and SMMPE only

$$\sum_{j=1}^{t} (p_j + 1) = \sum_{j=1}^{t} \omega_j \leqslant N \tag{3.11}$$

and

$$a_{ji}, \quad 0 \leqslant i \leqslant p_j, \quad 1 \leqslant j \leqslant t, \qquad \text{are linearly independent.} \tag{3.12}$$

Note that (3.12) is possible as the number of the vectors a_{ji} is at most N by (3.11).

As a consequence of (3.12) we obtain the following result:

LEMMA 3.3. *If* (3.12) *is satisfied, then the vectors* $\tilde{a}_{jl}$, $0 \leqslant l \leqslant p_j$, $1 \leqslant j \leqslant t$, *are linearly independent.*

Proof. It is enough to show that for each j the vectors $\tilde{a}_{jl}$, $0 \leqslant l \leqslant p_j$, are linearly independent. If we rewrite (3.6) in the form

$$\tilde{a}_{jl} = \sum_{i=0}^{p_j} \alpha_{li} a_{ji}, \qquad 0 \leqslant l \leqslant p_j, \tag{3.13}$$

where $\alpha_{li} = \binom{i}{i-l}$, then the matrix $[\alpha_{li}]_{l,i=0}^{p_j}$ is nonsingular. The result now follows from (3.12). ∎

We emphasize that the assumptions in (3.11) and (3.12) that we make on the function $F(z)$ are important only for SMPE and SMMPE procedures. For the approximation procedure STEA, (3.11) and (3.12) are not needed.

Although the assumptions in (3.11) and (3.12) pertain to the approximation procedures SMPE and SMMPE, they nevertheless involve only the function $F(z)$. We now make additional assumptions that are related more to the approximation procedures as they apply to $F(z)$. We assume throughout that

$$\begin{vmatrix} (q_1, a_{10}) & \cdots & (q_1, a_{1p_1}) & \cdots & (q_1, a_{t0}) & \cdots & (q_1, a_{tp_t}) \\ \vdots & & \vdots & & \vdots & & \vdots \\ (q_k, a_{10}) & \cdots & (q_k, a_{1p_1}) & \cdots & (q_k, a_{t0}) & \cdots & (q_k, a_{tp_t}) \end{vmatrix} \neq 0 \quad \text{for SMMPE,} \tag{3.14}$$

where $k = \sum_{j=1}^{t} \omega_j$, and that

$$\prod_{j=1}^{t} (q, a_{jp_j}) \neq 0 \qquad \text{for STEA.} \tag{3.15}$$

No additional assumption is needed for SMPE.

In order for (3.14) to hold it is necessary (but not sufficient) that the two sets of vectors $\{a_{ji}: 0 \leqslant i \leqslant p_j,\ 1 \leqslant j \leqslant t\}$ and $\{q_1, ..., q_k\}$, each be linearly independent, as has already been assumed.

4. Main Results

Our first result concerns the denominator polynomial $Q_{n,k}(z)$ of $F_{n,k}(z)$ and its zeros for $n \to \infty$ and is an analogue of the generalized Koenig theorem for Padé approximations and of Theorem 3.1 in [Si4]. The notation is identical to the one introduced in Section 3.

THEOREM 4.1. *Assume that the vector-valued function $F(z)$ is as described in the first paragraph of Section 3, and that, for the approximation procedures SMPE and SMMPE, $F(z)$ satisfies (3.11) and (3.12), in addition. Assume furthermore that (3.14) and (3.15) are satisfied for SMMPE and STEA, respectively. Then, provided*

$$k = \sum_{j=1}^{t} \omega_j, \tag{4.1}$$

the polynomial $Q_{n,k}(z) = \sum_{j=0}^{k} c_j^{(n,k)} z^{k-j}$, $c_k^{(n,k)} = 1$, exists for all sufficiently large n, and satisfies

$$Q_{n,k}(z) = \prod_{j=1}^{t} (1 - \lambda_j z)^{\omega_j} + O\left(\left|\frac{z_t}{R_1}\right|^n\right) \quad as \quad n \to \infty, \tag{4.2}$$

where $R_1 \in (|z_t|, R)$, but R_1 is arbitrary otherwise. Consequently, $Q_{n,k}(z)$, for $n \to \infty$, has ω_j zeros $z_{jl}(n)$, $1 \leqslant l \leqslant \omega_j$, that tend to z_j, $j = 1, 2, \ldots, t$. For each j and l we have

$$\limsup_{n \to \infty} |z_{jl}(n) - z_j|^{1/n} \leqslant \left|\frac{z_j}{R}\right|^{1/\omega_j}. \tag{4.3}$$

Let us denote

$$\hat{z}_j(n) = \frac{1}{\omega_j} \sum_{l=1}^{\omega_j} z_{jl}(n). \tag{4.4}$$

Then

$$\limsup_{n \to \infty} |\hat{z}_j(n) - z_j|^{1/n} \leqslant \left|\frac{z_j}{R}\right|. \tag{4.5}$$

Also the p_jth derivative of $Q_{n,k}(z)$ has exactly one zero $\tilde{z}_j(n)$ that tends to z_j and satisfies

$$\limsup_{n \to \infty} |\tilde{z}_j(n) - z_j|^{1/n} \leqslant \left|\frac{z_j}{R}\right|. \tag{4.6}$$

In case the function $F(z)$ has only polar singularities on the circle $\partial K = \{z : |z| = R\}$, the results in (4.2), (4.3), (4.5), and (4.6) *can be strengthened to read precisely*

$$Q_{n,k}(z) = \prod_{j=1}^{t} (1 - \lambda_j z)^{\omega_j} + O(\varepsilon(n)) \qquad as \quad n \to \infty, \tag{4.7}$$

$$z_{jl}(n) - z_j = O(\delta_j(n)^{1/\omega_j}) \qquad as \quad n \to \infty, \tag{4.8}$$

$$\hat{z}_j(n) - z_j = O(\delta_j(n)) \qquad as \quad n \to \infty, \tag{4.9}$$

and

$$\tilde{z}_j(n) - z_j = O(\delta_j(n)) \qquad as \quad n \to \infty, \tag{4.10}$$

respectively, where

$$\varepsilon(n) = n^{\alpha} \left|\frac{z_t}{R}\right|^n \qquad and \qquad \delta_j(n) = n^{\bar{p}} \left|\frac{z_j}{R}\right|^n. \tag{4.11}$$

Here α is some nonnegative integer, and $\bar{p}+1$ is the maximum of the multiplicities of the poles lying on ∂K. Also, if the poles whose moduli are $|z_t|$ are simple, then $\alpha = \bar{p}$. Finally, let us define $\lambda \equiv 1/z$, $\lambda_{jl}(n) \equiv 1/z_{jl}(n)$, and set

$$\hat{\lambda}_j(n) = \frac{1}{\omega_j} \sum_{l=1}^{\omega_j} \lambda_{jl}(n). \tag{4.12}$$

Then $\lambda_{jl}(n)$, $1 \leqslant l \leqslant \omega_j$, are the zeros of the polynomial (in λ) $\hat{Q}_{n,k}(\lambda) \equiv z^{-k} Q_{n,k}(z)$ that tend to λ_j. Similarly, the p_jth derivative of $\hat{Q}_{n,k}(\lambda)$ has a unique zero $\tilde{\lambda}_j(n)$ that tends to $\lambda_j = 1/z_j$ as $n \to \infty$. The results in (4.3), (4.5), *and* (4.6), *and in* (4.8)–(4.10) *hold also when the z's on the left-hand sides are replaced by the corresponding λ's.*

Proof. We do not intend to give the proofs of all of the results stated above. We are content with a short sketch of the proof of (4.2) and refer the reader to the appropriate references for the rest.

We start by observing that it is sufficient to analyze the determinant $D(\lambda^0, \lambda^1, ..., \lambda^k) = z^{-k} D(z^k, z^{k-1}, ..., z^0)$, which is proportional to $\hat{Q}_{n,k}(\lambda)$, which, in turn, is proportional to $Q_{n,k}(z)$. Employing Lemma 3.1, we obtain the asymptotic behavior of the u_{ij} in (1.5) for SMPE, SMMPE, and STEA. Substituting this in the determinant representation of (2.4) and following [SiB, Sect. 5.3], we obtain, for SMPE,

$$D(\lambda^0, \lambda^1, ..., \lambda^k) = W \left| \prod_{j=1}^{t} \lambda_j^{\omega_j} \right|^{2n} \left[\prod_{j=1}^{t} (\lambda - \lambda_j)^{\omega_j} + O\left(\left|\frac{\xi}{\lambda_t}\right|^n\right)\right] \qquad \text{as} \quad n \to \infty, \tag{4.13}$$

ξ being as in (3.7). Here

$$W = (-1)^k \hat{Z} \left| \prod_{j=1}^{t} \lambda_j^{p_j(p_j+1)/2} \prod_{1 \leq i < j \leq t} (\lambda_j - \lambda_i)^{\omega_i \omega_j} \right|^2, \tag{4.14}$$

where $\hat{Z}$ is the Gram determinant of the k vectors $\tilde{a}_{jl}$, $0 \leq l \leq p_j$, $1 \leq j \leq t$. By the assumption (3.12), Lemma 3.3 holds, so that these vectors are linearly independent. Consequently, $\hat{Z} > 0$; hence $W \neq 0$ for SMPE. For SMMPE and STEA, following [SiB, Sect. 5.1 and 5.2], we obtain

$$D(\lambda^0, \lambda^1, ..., \lambda^k) = W \left(\prod_{j=1}^{t} \lambda_j^{\omega_j} \right)^n \left[\prod_{j=1}^{t} (\lambda - \lambda_j)^{\omega_j} + O\left(\left| \frac{\xi}{\lambda_t} \right|^n \right) \right] \quad \text{as} \quad n \to \infty. \tag{4.15}$$

Here

$$W = (-1)^k \hat{Z} \left(\prod_{j=1}^{t} \lambda_j^{p_j(p_j+1)/2} \right) \left(\prod_{1 \leq i < j \leq t} (\lambda_j - \lambda_i)^{\omega_i \omega_j} \right), \tag{4.16}$$

where $\hat{Z}$ is the $k \times k$ determinant

$$\hat{Z} = \begin{vmatrix} z_{10,1} & \cdots & z_{1p_1,1} & \cdots & z_{t0,1} & \cdots & z_{tp_t,1} \\ z_{10,2} & \cdots & z_{1p_1,2} & \cdots & z_{t0,2} & \cdots & z_{tp_t,2} \\ \vdots & & \vdots & & \vdots & & \vdots \\ z_{10,k} & \cdots & z_{1p_1,k} & \cdots & z_{t0,k} & \cdots & z_{tp_t,k} \end{vmatrix}, \tag{4.17}$$

with

$$z_{jl,h} = (q_h, \tilde{a}_{jl}) \tag{4.18}$$

for SMMPE, and

$$z_{jl,h} = \sum_{i=l}^{p_j} (q, \tilde{a}_{ji}) \binom{h-1}{i-l} \lambda_j^{h-1} \tag{4.19}$$

for STEA. Performing elementary column transformations on (4.17), it can be shown that $\hat{Z}$ for SMMPE is equal to the determinant in (3.14), while $\hat{Z}$ for STEA is equal to

$$(-1)^{\sum_{j=1}^{t} p_j(p_j+1)/2} \left(\prod_{j=1}^{t} (q, a_{jp_j})^{\omega_j} \right) \left(\prod_{j=1}^{t} \lambda_j^{p_j(p_j+1)/2} \right) \left(\prod_{1 \leq i < j \leq t} (\lambda_j - \lambda_i)^{\omega_i \omega_j} \right).$$

By what has been assumed, we see that $\hat{Z} \neq 0$; hence $W \neq 0$ for SMMPE and STEA.

By $W \neq 0$, the proof of (4.2) is now complete for SMPE, SMMPE, and STEA. The rest of the theorem can be proved by employing the techniques that were used in the proof of Theorem 3.1 of [Si4]. ▮

Remarks. 1. The results of Theorem 4.1 seem to be best possible, in general. For one important special case, however, the results pertaining to the SMPE and STEA approximations can be improved considerably: In this special case the function $F(z)$ is rational with simple poles, and its residues form an orthogonal set of vectors with respect to the inner product $(\cdot, \cdot)$. We postpone the presentation of these results to Theorem 4.5 at the end of this section. Note also that, unlike those in (4.2), (4.3), (4.5), and (4.6), the results in (4.7)–(4.10) are truly asymptotic, and it is this that makes them superior to the former.

2. As can be seen from (4.3) and (4.8), if z_r and z_s are two poles in the open disk $K = \{z : |z| < R\}$ satisfying $|z_r| = |z_s|$, whose multiplicities ω_r ad ω_s satisfy $\omega_r < \omega_s$, then the approximations $z_{rl}(n)$, $1 \leqslant l \leqslant \omega_r$, that converge to z_r, do so faster than the approximations $z_{sl}(n)$, $1 \leqslant l \leqslant \omega_s$, that converge to z_s. Best rates of convergence are obtained for simple poles, i.e., those that have multiplicities equal to 1. In this sense, simple poles z_j have approximations $z_{j1}(n)$ that converge "optimally." Again, in this sense, the approximations $\hat{z}_j(n)$ and $\tilde{z}_j(n)$ to the multiple pole z_j have "optimal" convergence rates as can be seen from (4.5) and (4.6), whereas the individual $z_{jl}(n)$, $1 \leqslant l \leqslant \omega_j$, do not.

3. The proofs of the nth root asymptotic results in (4.5) and (4.6) can also be done by employing (4.2) and Theorem 4.2 as explained in the paragraph that follows the proof of Theorem 4.4.

Our next result concerns the convergence of $F_{n,k}(z)$ for $n \to \infty$ and is an analogue of de Montessus's theorem for Padé approximants and of Theorem 3.3 in [Si4]. Below and in the remainder of this work we use $|f|$ to also mean the norm of f in case $f \in C^N$.

THEOREM 4.2. *Let $F(z)$ and $F_{n,k}(z)$ be exactly as in Theorem 4.1. Then, as $n \to \infty$, $F_{n,k}(z)$ converges to $F(z)$ uniformly in any compact subset of $K \backslash \{z_1, ..., z_t\}$. In fact,*

$$F_{n,k}^{(r)}(z) - F^{(r)}(z) = O\left(\left|\frac{z}{R_1}\right|^n\right) \quad \text{as} \quad n \to \infty, \quad r = 0, 1, 2, ..., \tag{4.20}$$

uniformly in any compact subset of $K \backslash \{z_1, ..., z_t\}$, R_1 being, as in Theorem 4.1, in $(|z_t|, R)$, but arbitrary otherwise. In case $F(z)$ has only polar *singularities on* $\partial K = \{ : |z| = R\}$, *this result for* $r=0$ *can be improved to read precisely*

$$F_{n,k}(z) - F(z) = O\left(n^{\bar{p}}\left|\frac{z}{R}\right|^n\right) \quad as \quad n \to \infty, \tag{4.21}$$

uniformly in any compact subset of $K \setminus \{z_1, ..., z_t\}$, *where* $\bar{p}+1$ *is the maximum of the multiplicities of the poles of* $F(z)$ *that lie on* ∂K.

Proof. We start by noting that

$$\begin{aligned} F_{n,k}(z) - F(z) &= \frac{D(\sigma_0, \sigma_1, ..., \sigma_k)}{D(z^k, z^{k-1}, ..., z^0)} \equiv \frac{E_{n,k}(z)}{D(z^k, ..., z^0)}, \\ \text{with} \quad \sigma_j &= z^{k-j}[F_{n+\nu+j}(z) - F(z)], \qquad 0 \leqslant j \leqslant k. \end{aligned} \tag{4.22}$$

Next, by employing (3.5) and (3.8) in the determinant representation of $E_{n,k}(z)$, and following [SiB, Sect. 5.3], we obtain, for SMPE,

$$E_{n,k}(z) = O\left(\left|\prod_{j=1}^{t} \lambda_j^{\omega_j}\right|^{2n} \xi^n z^n\right) \quad \text{as} \quad n \to \infty, \tag{4.23}$$

ξ being as in (3.7). The result in (4.20) with $r=0$ for SMPE now follows by combining (4.23) and (4.13) in (4.22). For SMMPE and STEA, following [SiB, Sect. 5.1 and 5.2], we obtain

$$E_{n,k}(z) = O\left(\left|\prod_{j=1}^{t} \lambda_j^{\omega_j}\right|^{n} \xi^n z^n\right) \quad \text{as} \quad n \to \infty. \tag{4.24}$$

The result in (4.20) with $r=0$ for SMMPE and STEA now follows by combining (4.24) and (4.15) in (4.22). Both (4.23) and (4.24) hold uniformly in any compact subset of $K \setminus \{z_1, ..., z_t\}$ since the functions $b_{jl}(z)$ and $b(m, \xi, z)$ are all uniformly bounded away from the poles $z_1, ..., z_t$. Consequently, (4.20) holds uniformly in any compact subset of $K \setminus \{z_1, ..., z_t\}$. The proof of (4.21) can be achieved similarly. With (4.20) proved for $r=0$, the proof of (4.20) for $r \geqslant 1$ can be achieved by using Cauchy's formulas for expressing the derivatives of $F(z)$ and $F_{n,k}(z)$ as contour integrals. We have

$$F^{(r)}(z) = \frac{r!}{2\pi i} \int_{|\zeta - z| = \varepsilon} \frac{F(\zeta)}{(\zeta - z)^{r+1}} \, d\zeta \tag{4.25}$$

and, for n sufficiently large,

$$F_{n,k}^{(r)}(z)=\frac{r!}{2\pi i}\int_{|\zeta-z|=\varepsilon}\frac{F_{n,k}(\zeta)}{(\zeta-z)^{r+1}}\,d\zeta, \tag{4.26}$$

where z is a point of analyticity of $F(\zeta)$ and the disk $|\zeta-z|\leqslant\varepsilon$ contains no poles of $F(z)$. Since all the poles of $F_{n,k}(z)$ tend to those of $F(z)$ as $n\to\infty$, we see that, for sufficiently large n, the disk $|\zeta-z|\leqslant\varepsilon$ contains no poles of $F_{n,k}(z)$, so that (4.26) is correct. Now, taking the difference between (4.25) and (4.26), and taking norms of both sides, we obtain

$$\begin{aligned}|F_{n,k}^{(r)}(z)-F^{(r)}(z)| &\leqslant\frac{r!}{2\pi}\int_{|\zeta-z|=\varepsilon}\frac{|F_{n,k}(\zeta)-F(\zeta)|}{|\zeta-z|^{r+1}}\,|d\zeta|\\ &=\frac{r!}{\varepsilon^r}\max_{|\zeta-z|=\varepsilon}|F_{n,k}(\zeta)-F(\zeta)|.\end{aligned} \tag{4.27}$$

From (4.20) with $r=0$ we have

$$\max_{|\zeta-z|=\varepsilon}|F_{n,k}(\zeta)-F(\zeta)|=O\left(\left[\frac{|z|+\varepsilon}{R_1}\right]^n\right)\quad\text{as}\quad n\to\infty, \tag{4.28}$$

uniformly in any compact subset of $K\backslash\{z_1,\ldots,z_t\}$ in the ζ-plane. Substituting (4.28) in (4.27), and taking the lim sup of the nth root of both sides, we obtain

$$\limsup_{n\to\infty}|F_{n,k}^{(r)}(z)-F^{(r)}(z)|^{1/n}\leqslant\frac{|z|+\varepsilon}{R_1}. \tag{4.29}$$

Noting that $\varepsilon>0$ is arbitrary just as R_1 is, we conclude that (4.20) holds. ∎

Our next result shows how $F_{n,k}(z)$ can be used to construct approximations to the principal part of the Laurent expansion of $F(z)$ about any of the poles $z_1,\ldots,z_t$ with "optimal" accuracy, i.e., with the accuracy enjoyed by the $\hat{z}_j(n)$ and $\tilde{z}_j(n)$.

Let us rewrite (3.3) in the form

$$F(z)=\sum_{j=1}^{t}\sum_{i=0}^{p_j}\frac{d_{ji}}{(z-z_j)^{i+1}}+G(z), \tag{4.30}$$

where

$$d_{ji}=(-z_j)^{i+1}\,a_{ji}\qquad\text{for all } j, i. \tag{4.31}$$

THEOREM 4.3. *Let $F(z)$ and $F_{n,k}(z)$ be exactly as in Theorem* 4.1. *Denote*

$$\zeta_j(n)=\hat{z}_j(n) \quad or \quad \zeta_j(n)=1/\hat{\lambda}_j(n)$$
$$or \quad \zeta_j(n)=\tilde{z}_j(n) \quad or \quad \zeta_j(n)=1/\tilde{\lambda}_j(n), \tag{4.32}$$

and denote the residue of the rational function $(z-\zeta_j(n))^i F_{n,k}(z)$ at $z_{jl}(n)$ by $d_{ji,l}(n)$, $1\leqslant l\leqslant \omega_j$, $0\leqslant i\leqslant p_j$. Then

$$\limsup_{n\to\infty}\left|\sum_{l=1}^{\omega_j} d_{ji,l}(n)-d_{ji}\right|^{1/n}\leqslant\left|\frac{z_j}{R}\right|. \tag{4.33}$$

Proof. Let $K_j=\{z:|z-z_j|\leqslant\varepsilon\}$ with $\varepsilon>0$ chosen sufficiently small to ensure that K_j contains only z_j and no other poles of $F(z)$. By Theorem 4.1, for n sufficiently large, K_j contains only $z_{jl}(n)$, $1\leqslant l\leqslant\omega_j$, and no other poles of $F_{n,k}(z)$. Let ∂K_j denote the boundary of K_j. By the fact that (4.20) holds uniformly in any compact subset of $K\backslash\{z_1,...,z_t\}$, we have

$$\max_{z\in\partial K_j}|F_{n,k}(z)-F(z)|=O\left(\left[\frac{|z_j|+\varepsilon}{R_1}\right]^n\right) \quad \text{as} \quad n\to\infty, \tag{4.34}$$

R_1 being as in Theorems 4.1 and 4.2. We now note that

$$d_{ji}=\frac{1}{2\pi\mathrm{i}}\int_{\partial K_j}(z-z_j)^i F(z)\,dz \tag{4.35}$$

and, for n sufficiently large,

$$\sum_{l=1}^{\omega_j} d_{ji,l}(n)=\frac{1}{2\pi\mathrm{i}}\int_{\partial K_j}(z-\zeta_j(n))^i F_{n,k}(z)\,dz. \tag{4.36}$$

Consequently,

$$\sum_{l=1}^{\omega_j} d_{ji,l}(n)-d_{ji}=\frac{1}{2\pi\mathrm{i}}\int_{\partial K_j}\Delta_{n,k}(z)\,dz, \tag{4.37}$$

where

$$\Delta_{n,k}(z)=(z-\zeta_j(n))^i F_{n,k}(z)-(z-z_j)^i F(z)\,dz. \tag{4.38}$$

Now

$$|\Delta_{n,k}(z)|\leqslant|z-\zeta_j(n)|^i\,|F_{n,k}(z)-F(z)|$$
$$+|(z-\zeta_j(n))^i-(z-z_j)^i|\,|F(z)|. \tag{4.39}$$

The first term on the right-hand side of (4.39) is $O([(|z_j|+\varepsilon)/R_1]^n)$ as $n\to\infty$, by (4.34). Also,

$$(z-\zeta_j(n))^i-(z-z_j)^i=(z_j-\zeta_j(n))\sum_{m=0}^{i-1}(z-\zeta_j(n))^m(z-z_j)^{i-1-m}$$
$$=O\left(\left[\left|\frac{z_j}{R}\right|+\varepsilon_1\right]^n\right) \quad \text{as} \quad n\to\infty,\ \varepsilon_1>0 \text{ arbitrary,} \tag{4.40}$$

by Theorem 4.1, cf. (4.5) and (4.6). Combining (4.34) and (4.40) in (4.39), we obtain

$$\limsup_{n\to\infty}\left(\max_{z\in\partial K_j}|\Delta_{n,k}(z)|\right)^{1/n}\leqslant\left|\frac{z_j}{R}\right|. \tag{4.41}$$

Taking the modulus of both sides of (4.37), and employing (4.41), (4.33) now follows. ∎

It is interesting to note that the technique used in the proof of Theorem 4.3 can be used to prove other useful results concerning $F(z)$. One such result is given in Theorem 4.4 below.

THEOREM 4.4. *Let $F(z)$ and $F_{n,k}(z)$ be as in Theorem 4.1, and let $H(z)$ be analytic in a neighborhood of z_j. Then*

$$\limsup_{n\to\infty}\left|\frac{1}{\omega_j}\sum_{l=1}^{\omega_j}H(z_{jl}(n))-H(z_j)\right|^{1/n}\leqslant\left|\frac{z_j}{R}\right|. \tag{4.42}$$

Proof. Let the disk K_j in the proof of Theorem 4.3 be so small that it does not include any of the zeros of $F(z)$. Then there is a constant vector $w\in C^N$ for which $(w,F(z))\neq 0$ on ∂K_j. For instance, $w=a_{jp_j}$ serves this purpose when the radius of K_j is sufficiently small. Thus

$$-\omega_j H(z_j)=\frac{1}{2\pi i}\int_{\partial K_j}H(z)\frac{(w,F'(z))}{(w,F(z))}dz \tag{4.43}$$

and

$$-\sum_{l=1}^{\omega_j}H(z_{jl}(n))=\frac{1}{2\pi i}\int_{\partial K_j}H(z)\frac{(w,F'_{n,k}(z))}{(w,F_{n,k}(z))}dz. \tag{4.44}$$

The rest of the proof can now be completed as that of Theorem 4.3. ∎

It is easy to see that, letting $H(z)=z$ in (4.42), we obtain the result in (4.5) that pertains to $\hat{z}_j(n)$. Similarly, letting $H(z)=z^{-1}$ in (4.42), we obtain the analogous result pertaining to $\hat{\lambda}_j(n)$. The stronger result for

$\hat{z}_j(n)$ given in (4.9), and the corresponding result for $\hat{\lambda}_j(n)$, cannot be obtained from Theorem 4.4, however. Finally, letting $H(z)=z$ and replacing $F(z)$ and ω_j by $F^{(\omega_j-1)}(z)$ and 1, respectively, in (4.43) and (4.44), we can obtain the result in (4.6) pertaining to $\tilde{z}_j(n)$.

We now give the improved version of Theorem 4.1 for the special case mentioned in Remark 1 following the proof of Theorem 4.1 that pertains to the SMPE and STEA procedures.

THEOREM 4.5. *Let the vector-valued function $F(z)$ in Theorem 4.1 be rational with simple poles, i.e.,*

$$F(z)=\sum_{j=1}^{\mu}\frac{a_j}{1-\lambda_j z}+G(z), \tag{4.45}$$

where a_j are constant vectors in C^N and $G(z)$ is a vector-valued polynomial. Assume furthermore that $\mu \leqslant N$ and that

$$(a_j, a_h)=0 \qquad \text{if} \quad j \neq h. \tag{4.46}$$

Also, in the definition of the STEA procedure, replace the fixed vector q in (1.5) by u_n. Then, provided

$$|z_k|<|z_{k+1}| \qquad \text{or equivalently} \qquad |\lambda_k|>|\lambda_{k+1}|, \tag{4.47}$$

the polynomials $Q_{n,k}(z)$ associated with the SMPE and STEA approximation procedures exist for all n sufficiently large and satisfy

$$Q_{n,k}(z)=\prod_{j=1}^{k}(1-\lambda_j z)+O\left(\left|\frac{z_k}{z_{k+1}}\right|^{2n}\right) \qquad \text{as} \quad n\to\infty. \tag{4.48}$$

Consequently, both for the SMPE and STEA procedures, $Q_{n,k}(z)$, for $n\to\infty$, has exactly one zero $z_j(n)$ that tends to z_j, $j=1, 2, ..., k$. For each j, $1\leqslant j\leqslant k$, we have precisely

$$z_j(n)-z_j=O\left(\left|\frac{z_j}{z_{k+1}}\right|^{2n}\right) \qquad \text{as} \quad n\to\infty, \tag{4.49}$$

and this holds true also when $z_j(n)$ and z_j on the left-hand side are replaced by $\lambda_j(n)=1/z_j(n)$ and $\lambda_j=1/z_j$.

Proof. We start by observing that, from (4.45) and the fact that $G(z)$ is a polynomial,

$$u_m=\sum_{j=1}^{\mu}a_j\lambda_j^m \qquad \text{for all sufficiently large } m. \tag{4.50}$$

The result for the SMPE procedure now follows from Theorem 2.1 in [Si3], which, in turn, is a corollary of Lemmas 2.2 and 2.3 there. The result for the STEA procedure can be obtained in exactly the same way as a corollary of Lemmas 2.2 and 2.3 in [Si3] once we observe that $u_{ij} = (u_{n+i}, u_{n+j})$ for SMPE and $u_{ij} = (u_n, u_{n+i+j})$ for STEA, under the assumption (4.46), have almost identical structure, namely,

$$u_{ij}^{\mathrm{SMPE}} = \sum_{s=1}^{\mu} [(a_s, a_s)\,\bar{\lambda}_s^i]\,\lambda_s^j\,|\lambda_s|^{2n}$$

and

$$u_{ij}^{\mathrm{STEA}} = \sum_{s=1}^{\mu} [(a_s, a_s)\lambda_s^i]\lambda_s^j\,|\lambda_s|^{2n}$$

for all sufficiently large n. For the SMPE procedure, Theorem 2.1 in [Si3] also provides the precise asymptotic constant that multiplies $|z_j/z_{k+1}|^{2n}$ in (4.49) in terms of the λ_i and the a_i. The asymptotic constant for the STEA procedure can be obtained in a similar fashion. ∎

It is important to note that the condition given in (4.1) for Theorems 4.1 and 4.2 seems to be crucial as far as the convergence of $Q_{n,k}(z)$ and $F_{n,k}(z)$ is concerned. According to this condition, k is the precise number of poles, counted according to their multiplicities, contained in K. When this condition is not satisfied, i.e., when k is larger than this number, we cannot expect convergence to take place, in general. When the only singularities on ∂K are poles, however, it might be possible to obtain some convergence results under certain conditions when k is greater than the number of poles in K but smaller than the number of poles in $K \cup \partial K$. Interesting results for this problem pertaining to classical Padé approximants have been given in [Si4, Sect. 6]. Analogous results that form an extension of Theorem 4.5 have been provided in [Si3, Sect. 3]. We do not intend to go into this matter in the present work, however.

5. Further Results on Residues

From the way the denominator polynomial $Q_{n,k}(z)$ of $F_{n,k}(z)$ is constructed, it is clear that the approximations $\hat{z}_j(n)$ and $\tilde{z}_j(n)$ to $z_j, j = 1, 2, ..., t$, are obtained from the vectors $u_n, u_{n+1}, ..., u_{n+k}$ in the case of the SMPE and SMMPE procedures and from $u_n, u_{n+1}, ..., u_{n+2k-1}$ in the case of the STEA procedure. This means that the vectors $u_0, u_1, ..., u_{n-1}$ need not be saved if we are only interested in approximating the smallest poles of $F(z)$. This is important since we are considering the limiting process in which $n \to \infty$.

From (4.30), (4.31), and Theorem 4.3, the approximation to the vector a_{ji} is given as $a_{ji}(n) = (-\lambda_j)^{i+1} \sum_{l=1}^{\omega_j} d_{ji,l}(n)$. We now show that the computation of the vectors $a_{ji}(n)$ can be made to enjoy the same property in the sense that knowledge of $u_0, u_1, \ldots, u_{n-1}$ is not essential in this case either.

Let us write the vectors $d_{ji,l}(n)$ introduced in Theorem 4.3 explicitly. By the fact that $z_{jl}(n)$ is a simple pole of $F_{n,k}(z)$ for n sufficiently large, we have

$$d_{ji,l}(n) = (z - \zeta_j(n))^i \left. \frac{\sum_{r=0}^{k} c_r^{(n,k)} z^{k-r} F_{n+\nu+r}(z)}{\sum_{r=0}^{k} c_r^{(n,k)} (k-r)\, z^{k-r-1}} \right|_{z = z_{jl}(n)}. \tag{5.1}$$

Writing $F_{n+\nu+r}(z) = F_{n+\nu}(z) + \sum_{m=n+\nu+1}^{n+\nu+r} u_m z^m$ in (5.1), and using the fact that $\sum_{r=0}^{k} c_r^{(n,k)} z^{k-r} |_{z = z_{jl}(n)} = 0$, we obtain

$$d_{ji,l}(n) = (z - \zeta_j(n))^i \left. \frac{\sum_{r=0}^{k} c_r^{(n,k)} z^{k-r} \sum_{m=n+\nu+1}^{n+\nu+r} u_m z^m}{\sum_{r=0}^{k} c_r^{(n,k)} (k-r)\, z^{k-r-1}} \right|_{z = z_{jl}(n)}, \tag{5.2}$$

in which the absence of the vectors $u_0, u_1, \ldots, u_{n-1}$ is transparent.

Developing the approach that leads to (5.2) further, in the remainder of this section we give approximants to the a_{ji} that are different from the $a_{ji}(n)$ above. These new approximations are used in the treatment of the matrix eigenvalue problem in [Si6].

Consider the meromorphic function

$$\hat{F}(z) = \frac{F(z) - F_{n+\nu}(z)}{z^{n+\nu+1}}, \tag{5.3}$$

which is analytic at $z = 0$ and has the Maclaurin series expansion

$$\hat{F}(z) = \sum_{i=0}^{\infty} u_{n+\nu+i+1} z^i. \tag{5.4}$$

By (4.30) we can write

$$\hat{F}(z) = \sum_{i=0}^{p_j} \frac{\hat{d}_{ji}}{(z - z_j)^{i+1}} + \hat{G}_j(z), \tag{5.5}$$

where

$$\hat{d}_{ji} = z_j^{-n-\nu-1} \sum_{l=i}^{p_j} \binom{-n-\nu-1}{l-i} z_j^{-l+i} d_{jl}, \tag{5.6}$$

and $\hat{G}_j(z)$ is analytc at z_j. Consequently,

$$\hat{d}_{ji} = \frac{1}{2\pi i} \int_{\partial K_j} (z - z_j)^i \hat{F}(z)\, dz, \tag{5.7}$$

with the disk K_j as in the proof of Theorem 4.3.

Consider next the rational function

$$\hat{F}_{n,k}(z) = \frac{F_{n,k}(z) - F_{n+\nu}(z)}{z^{n+\nu+1}}. \tag{5.8}$$

Denote the residue of $(z - \zeta_j(n))^i \hat{F}_{n,k}(z)$ at $z_{jl}(n)$ by $\hat{d}_{ji,l}(n)$, $1 \leqslant l \leqslant \omega_j$, and let

$$\hat{d}_{ji}(n) = \sum_{l=1}^{\omega_j} \hat{d}_{ji,l}(n). \tag{5.9}$$

Then

$$\hat{d}_{ji}(n) = \frac{1}{2\pi i} \int_{\partial K_j} (z - \zeta_j(n))^i \hat{F}_{n,k}(z)\, dz \tag{5.10}$$

for n sufficiently large.

Subtracting (5.7) from (5.10), invoking (5.3) and (5.8), and observing that the contribution of the polynomial $F_{n+\nu}(z)$ is zero, we obtain

$$\hat{d}_{ji}(n) - \hat{d}_{ji} = \frac{1}{2\pi i} \int_{\partial K_j} \frac{\Delta_{n,k}(z)}{z^{n+\nu+1}}\, dz, \tag{5.11}$$

with $\Delta_{n,k}(z)$ as defined in (4.38). Following now the proof of Theorem 4.3, we obtain

$$\limsup_{n \to \infty} |\hat{d}_{ji}(n) - \hat{d}_{ji}|^{1/n} \leqslant R^{-1}. \tag{5.12}$$

It is important to note that

$$\begin{aligned} \hat{d}_{ji,l}(n) &= d_{ji,l}(n)/z_{jl}(n)^{n+\nu+1} \\ &= (z - \zeta_j(n))^i \left. \frac{\sum_{r=0}^{k} c_r^{(n,k)} z^{k-r} \sum_{m=1}^{r} u_{n+\nu+m} z^{m-1}}{\sum_{r=0}^{k} c_r^{(n,k)} (k-r)\, z^{k-r-1}} \right|_{z = z_{jl}(n)}, \end{aligned} \tag{5.13}$$

so that the factor $z_{jl}(n)^n$ has disappeared from $\hat{d}_{ji,l}(n)$. Consequently, $\hat{d}_{ji}(n)$ does not involve the factors $z_{jl}(n)^n$, $1 \leqslant l \leqslant \omega_j$. In view of the fact that we are interested in the limit as $n \to \infty$, the absence of the $z_{jl}(n)^n$ is expected to have a stabilizing effect in actual numerical computations. In fact, the developments of this section were prompted also by the desire to eliminate the factors $z_{jl}(n)^n$ from the approximations to the a_{ji}. We make wide use of these new approximations to the a_{ji} in our treatment of the matrix eigenvalue problem in [Si6].

6. Extension to General Linear Spaces

In the previous sections we considered $F(z)$ to be a function from $\boldsymbol{C}$ to $\boldsymbol{C}^N$. We would now like to explore the possibility that $F(z)$ is a function from $\boldsymbol{C}$ to a general linear space $\boldsymbol{B}$. We note that in [Si1, Si3, SiB, SiFSm] vector extrapolation methods were defined and analyzed for sequences in such spaces under certain assumptions.

There seems to be no problem in defining the rational approximation procedures SMPE, SMMPE, and STEA in general spaces. We only need to demand that $\boldsymbol{B}$ be

(i) an inner product space for SMPE,

(ii) a normed space for SMMPE and STEA.

Then if $F(z)$ is as given in (1.1) with $u_m \in \boldsymbol{B}$, $m = 0, 1, \ldots$, the rational approximations $F_{n,k}(z)$ to $F(z)$ are exactly as in (1.2) and (1.3). The scalars $c_j^{(n,k)}$ again satisfy the linear equations in (1.4) with u_{ij} defined by

$$u_{ij} = \begin{cases} (u_{n+i}, u_{n+j}) & \text{for SMPE,} \\ Q_{i+1}(u_{n+j}) & \text{for SMMPE,} \\ Q(u_{n+i+j}) & \text{for STEA.} \end{cases} \tag{6.1}$$

Here too $(\cdot, \cdot)$ is the inner product associated with the inner product space $\boldsymbol{B}$, whose homogeneity property is such that $(\alpha x, \beta y) = \bar{\alpha}\beta(x, y)$ for $x, y \in \boldsymbol{B}$ and $\alpha, \beta \in \boldsymbol{C}$. $Q_1, \ldots, Q_k$, and Q are bounded linear functionals on the normed space $\boldsymbol{B}$, and $Q_1, \ldots, Q_k$ are, of course, assumed to be linearly independent. In case $\boldsymbol{B}$ is a complete inner product space in addition to being a normed space, the functionals $Q_1, \ldots, Q_k$ and Q have unique representers $q_1, \ldots, q_k$, and q, respectively, in $\boldsymbol{B}$, so that u_{ij} for SMMPE and STEA become

$$u_{ij} = \begin{cases} (q_{i+1}, u_{n+j}) & \text{for SMMPE,} \\ (q, u_{n+i+j}) & \text{for STEA;} \end{cases} \tag{6.2}$$

cf. (1.5).

With $F_{n,k}(z)$ properly defined, we now go on to discuss the extension of Theorems 4.1–4.4. Going through the proofs of these theorems, we see that they hold when $F(z)$ satisfies the conditions stated in Section 3 provided (3.7) and (3.10) are interpreted in a suitable manner. Thus, (3.7) is interpreted as

$$|\tilde{a}(m, \xi)| \leqslant M(\xi) \equiv \max_{|z| = \xi^{-1}} |G(z)|, \qquad \xi^{-1} \in (|z_t|, R), \qquad \text{but arbitrary,} \tag{6.3}$$

while (3.10) is interpreted as

$$|b(m, \xi, z)| \leqslant M(\xi) \frac{|\xi z|}{1 - |\xi z|}, \qquad |z| < \xi^{-1}, \tag{6.4}$$

with $M(\xi)$ as in (6.3). As before, $|f|$ stands for the norm of f when $f \in \boldsymbol{B}$. In case $\boldsymbol{B}$ is an inner product space, this norm can be taken to be the one induced by the inner product. Also, in case $\boldsymbol{B}$ is only a normed linear space, in the proof of Theorem 4.4 the assertion $(w, F(z)) \neq 0$ on ∂K_j is replaced fy $T(F(z)) \neq 0$ on ∂K_j, where T is some bounded linear functional on $\boldsymbol{B}$.

The result given in Theorem 4.5 can be maintained both when $F(z)$ is a rational function and when $F(z)$ has an infinite number of poles so that the u_m satisfy asymptotically

$$u_m \sim \sum_{j=1}^{\infty} a_j \lambda_j^m \qquad \text{as} \quad m \to \infty. \tag{6.5}$$

Precisely this is the subject of [Si3].

Finally, we mention that one immediate application of the rational approximation procedures is to the solution of the operator equation

$$x = zAx + b, \tag{6.6}$$

where A is a bounded linear operator on $\boldsymbol{B}$. In this case the solution $x = (I - zA)^{-1} b$ to (6.6) has the convergent Maclaurin expansion

$$x = \sum_{m=0}^{\infty} u_m z^m \qquad \text{with} \quad u_m = A^m b, \quad m = 0, 1, ... \tag{6.7}$$

Under appropriate conditions on the spectrum of A all of the results of Sections 4 and 5 hold. We leave out the details.

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